The following is the Introduction to Albert Lautman's *Essai sur les notions de structure et d'existence en mathématiques: Les schémas de structure*. Paris, Hermann & Cle Ed., 1938. p. 7-15. Original translaion by Taylor Adkins on 10/16/07.

This book is born from the feeling that in the development of mathematics, a reality continues which mathematical philosophy has as a function to recognize and describe. The spectacle of the majority of the modern theories of mathematical philosophy is in this respect extremely disappointing. Generally, the analysis of mathematics reveals only very little things and very poor things, like the research of identity or the tautological character of propositions [1]. It is true that in Meyerson the application of the rational identity to various mathematics supposes a reality which resists identification; it seems that there is thus the indication that the nature of this reality is different from the too simplistic diagram with which one tries to describe it; on the other hand, the development of the concept of tautology has completely eliminated from Russell's school the idea of a reality suitable for mathematics.

For Wittgenstein and Carnap, mathematics is nothing more than a language indifferent to the contents that it expresses. Only empirical proposals would refer to an objective reality, and mathematics would be only a system of formal transformations making it possible to connect the ones to the other the data of physics. If one tries to understand the reasons for this progressive fading of mathematical reality, one can be brought to conclude that it results from the use of the deductive method. Wanting to build all the mathematical notions starting from a small number of notions and primitive logical propositions makes one lose sight of the qualitative and integral character of the constituted theories.

However, what this mathematics lets us hope for with the philosopher is a truth which would appear in the harmony of its edifices, and in this field as well as others, the research of the primitive notions must yield place to a synthetic study of the whole. It appears to us in this respect in a quite strange connection, which after having carried out the most complete investigations on theories relative to space and number, Poirier has wanted to see in mathematics only a set of meaningless symbols [2]. He seems to have approached these symbols with the intention of asking them to enrich the indications that suggest the reality of external perception or internal sense. Reality is for him before all else that of immediate experience, and abstract theories do not give us any grasp on it. Poirier almost reproaches these theories for the greatness of their perfection. The ease with which these theories correspond gives to the aspect of each one of them a possibly arbitrary character, as well as others. None of these impose themselves on the spirit the feeling of a necessity that would result from the nature of things, and one never finds anything but formal processes, which do not answer a "natural and intuitive classification" of their objects.

We believe that it is possible to arrive at less negative conclusions, and contemporary mathematical philosophy remains engaged on two different paths, each with a positive study of mathematical reality. This reality can indeed be characterized by the way in which it is allowed to be apprehended and organized; it can also be seen in an intrinsic way, from the point of view of its own structure. We first of all, we will try to quickly summarize the fundamental ideas of both methods.

There is no philosopher today who has developed more than Leon Brunschvicg the idea that the objectivity of mathematics was the work of the intelligence, in its effort to triumph over resistances that the matter on which it works opposes. This matter is neither easy nor uniform; it has its folds, its edges, its irregularities, and our designs are nothing but a provisional arrangement which permits the spirit to keep going. Mathematics was constituted like physics; in order to explain it, we have to examine the history of paradoxes that the progress of reflection has rendered understandable through a constant renewal of the sense of these essential notions. The irrational numbers, the infinitely small, the continuous functions without derivative, the transcendence of *e* and π, and the transfinite were admitted by an incomprehensible necessity of fact before one had a deductive theory of them. It was once believed that the fate of certain physical constants like *c* or *h* were essential in an incomprehensible way in the most divergent fields, until the genius of Maxwell, Planck or Einstein knew to see in the constancy of their value the connection of electricity and light, light and energy. One thus understands Brunschvicg's distrust with respect to all the attempts which would like to deduce the unit from mathematics starting from a small number of initial principles. Brunschvicg also opposed, in *Les Etapes de la philosophie mathématique*, the reduction of mathematics to logic, against the idea that there could be general principles in mathematics like Poncelet's principle of continuity or the principle of permanence in Hankel's formal laws. Any effort of deduction *a priori* tends for him to reverse the natural order of the spirit in mathematical discovery. One would however not have to interpret Brunschvicg's mathematical philosophy as a pure psychology of creative invention. "Between the adventures of the invention", he writes,

> which interests only one individual consciousness, and the forms of speech which relates to especially the pedagogical tradition, (mathematical philosophy) will delimit the ground where the collective acquisition of the knowledge occurred, it will recognize the royal roads that have traced there the creative intelligence.

Between the psychology of the mathematician and the logical deduction, there must be a place for an intrinsic characterization of reality. It is necessary that it takes part at the same time of the movement of the intelligence and logical rigor, without merging either with one or the other, and it will be our task to test this synthesis.

The structural point of view to which we must thus also refer to is that of the meta-mathematics of Hilbert. We know the difference that separates the Hilbertian design of mathematics from that of the Principia *Mathematica* of Russell and Whitehead. Hilbert substitutes for the method of genetic definitions that of axiomatic definitions, and far from wanting to rebuild the whole of mathematics starting from logic, he introduces, on the contrary, while passing from logic to arithmetic and from arithmetic to analysis, new variables and new axioms that widen the field of consequences each time. Here for example, according to Bernays who published in the edition of Hilbert's complete works an overall study of Hilbert's work on the foundations of mathematics, all that is necessary to be given to formalize arithmetic: the calculation of propositions, the axioms of equality, the arithmetic axioms of the function of the "following" (a + I), the equations of recurrence for addition and multiplication and finally a certain form of the axiom of choice. To formalize the analysis, it is necessary to be able to apply the axiom of choice, not only with numeric variables, but with a higher category of variables, in which the variables are functions of numbers. Mathematics thus arises as successive syntheses wherein each stage is irreducible to the former stage. Moreover, and this is the most important, a theory thus formalized is unable to bring with it the proof of its internal coherence; meta-mathematics should be superimposed as that which takes formalized mathematics as an object and studies it from the double point of view of non-contradiction and completion. The duality of plans that Hilbert thus establishes between formalized mathematics and the meta-mathematical study of this formalism has as a consequence the fact that the concepts of non-contradiction and completion govern a formalism inside of which they do not appear as concepts defined in this formalism. It is to express this role dominating meta-mathematical concepts compared to formalized mathematics that Hilbert [3] writes:

> Demonstrable axioms and propositions, i.e. the formulas which are born from the set of these reciprocal actions (namely the formal deduction and the addition of new axioms), are the images of the thoughts which constitute the ordinary processes of developed mathematics until now, but are not the truths in the absolute sense. Truths in the absolute sense are rather completely the views (*Einsichten*) which give my theory of demonstration that which concerns the resolvability and the non-contradiction of these systems of formulas.

Mathematical theory receives its value from the meta-mathematical properties that its structure thus incarnates.

The structural design and the dynamic design of mathematics first of all seem to oppose themselves: the one indeed tends to regard a mathematical theory as a completed whole, independent of time, the other on the contrary does not separate the temporal stages from its development; for the former, the theories are like beings qualitatively distinct from each other, while the latter sees in each one an infinite power of expansion beyond its limits and connection with the others, because it affirms itself as the unity of the intelligence. We would however like, in the following pages, to try to develop a design of mathematical reality where the fixity of logical notions is combined with the movement that lives through these theories.

In the meta-mathematics of Hilbert, one proposes to examine mathematical theories from the point of view of the logical concepts of non-contradiction and completion, but it is there only one ideal towards which research is directed, and one knows at what point this ideal actually seems difficult to attain [4]. One can thus consider meta-mathematics the idea of certain perfect structures, eventually realizable by effective mathematical theories (independently of knowing if there exist theories enjoying the properties in question, because one can merely have the statement of a logical problem without having by any means the mathematical means to resolve it. This distinction between the position of a logical problem and its mathematical solution has at times seemed hardly fertile, because what is essential is not to know whether a theory could be non-contradictory, but that it is able to decide effectively if it is or if it is not.

However, it has appeared possible to us to consider other logical notions, also likely to be possibly connected one to the other within a mathematical theory and which are such that, contrary to the preceding cases, the mathematical solutions of the problems which they pose can comprise an infinite number of degrees. Partial results, rapprochements stopped midway, tests which still resemble gropings, are organized under the unity of a common theme and let us see in their movement a connection which takes shape between certain abstract ideas, which we propose to call dialectical. Mathematics, and especially modern mathematics, algebra, group theory, topology [5], thus appeared to us to recount, mixed with constructions in which the mathematician is interested, another, more disguised history, and made for the philosopher.

A dialectical action is constantly played out in the background and it is in order to clarify this that our six chapters will converge on this point. The first three chapters deal especially with de-structured mathematical notions. We study in chapter I (the local and the global) the almost organic solidarity which pushes the parts to be organized in a whole and the whole to be reflected in them; we examine then in chapter II (Intrinsic properties and inductive properties) if it is possible to bring back for the relations that a mathematical being supports with an ambient milieu, in the characteristic inherent properties of this being. We show in chapter III (rise towards completion) how the structure of an imperfect being can sometimes preform the existence of a perfect being in which any imperfection has disappeared. Then the three chapters relating to the concept of existence come. We try to develop in chapter IV (Essence and Existence) a new theory of the relations of essence and existence where one sees the structure of a being to be interpreted in terms of existence for beings other than the being of which one studies the structure. Chapter V (the Mixed) describes certain intermediate Mixtures between different kinds of Beings and whose consideration is often necessary to operate the passage of one kind of being to another kind of being; our final chapter (Of the exceptional character of existence) finally describes the processes by which a being can be distinguished within an infinity from the others.

We would like to show that ideas that are at the head of each chapter and which appear to us to dominate the movement of certain mathematical theories, to be conceivable independently of mathematics, are nevertheless not likely to be addressed in a direct study. They exist only compared to a matter that they penetrate in the intelligence, but one can say that on the other hand that they are the ones that confer on mathematics its eminent philosophical value. The method that we will follow is thus primarily a descriptive method of analysis; the mathematical theories constitute for us a given in which we will endeavor to release the ideal reality in which this matter takes part.

1. Cf. this passage of Russell: “They (mathematical propositions) have all the characteristics that a moment ago it was advisable to call tautology. This combined with the fact that they can be

expressed using variables or logical constants, will provide the definition of logic or pure mathematics."

2. Cf. René Poirier, *Essais sur quelques caractères des notions d'espace et de temps*.

3. Hilbert, *Die logischen Grundlagen der Mathematik*

4. Cf. Jean Cavaillès, *Méthode axiomatique et formalisme. Essai sur le problème du fondement des mathématiques.*

5. We expose this in our complementary thesis: *Essai sur l'unité des sciences mathématiques dans leur développement actuel,* certain aspects that make it possible to distinguish modern mathematics from classical mathematics.